Midsemestral exam
September 2011
B.Math.(Hons.) IInd year
Algebra III — B.Sury
BE BRIEF!

Q 1. Let A be a commutative ring with unity. If $I \neq J$ are different maximal ideals, prove that
$$A/(IJ) \cong A/I \times A/J$$

OR

If M is a maximal ideal of $C[0,1]$, show that there exists $a \in [0,1]$ such that
$$M = \{f \in C[0,1] : f(a) = 0\}$$

Q 2.
(i) Let A be a commutative ring with unity. If the complement of the set of units is an ideal of A, prove that A is a local ring.
(ii) Using (i) or otherwise, deduce that $\mathbf{C}[[X]]$ is local.

OR

Let A be a commutative ring with unity. Let $f \in A[X]$ be so that fg is the zero polynomial for some non-zero polynomial $g \in A[X]$. Show that there exists $0 \neq a \in A$ such that af is the zero polynomial.

Q 3. Let A be a commutative ring with unity. Let
$$B = \{a \in A : ab = 0 \ for \ some \ b \neq 0\}$$
Show that there exists a prime ideal contained in B.

OR

Let A be a commutative ring with unity. call a prime ideal *'minimal'* if it does not contain any other prime ideal properly. Show that the set of nilpotent elements in A is the intersection of minimal prime ideals of A.

Q 4. Identify which among the following are integral domains and write down their quotient fields (without proof):
(a) $\mathbf{Z}[X]$
(b) the power set $P(\mathbf{N})$ of $\mathbf{N}$
(c) $S^{-1}\mathbf{Z}$, where $S = \{2^n : n \geq 0\}$

(d) $\mathbf{Z}_n$ for an arbitrary positive integer n
(e) $\mathbf{Z}[i][[X]]$
(f) $\mathbf{R}[t,u]/(tu-1)$

Q 5. Let a be a non-zero integer. Consider the subring

$$\mathbf{Z}[1/a] = \{\frac{u}{a^n} : u \in \mathbf{Z}, n \geq 0\}$$

of $\mathbf{Q}$. Prove that $\mathbf{Z}[1/a] \cong \mathbf{Z}[X]/(aX-1)$.

OR

If α is an algebraic integer (a complex number which satisfies a monic integral polynomial), then prove that there is a unique monic polynomial f with rational coefficients such that $f(\alpha) = 0$ and all polynomials $g \in \mathbf{Q}[X]$ satisfying $g(\alpha) = 0$ must divide f in $\mathbf{Q}[X]$. Further, show that f must be in $\mathbf{Z}[X]$.

Q 6. Define $N : \mathbf{Z}[i] \to \mathbf{Z}$ by $N(a+ib) = a^2 + b^2$. If $N(a+ib)$ is a prime number, prove that $a+ib$ is an irreducible element of $\mathbf{Z}[i]$.

OR

For a positive integer n, determine with proof the integer $a(n)$ such that $\mathbf{Z}[i]/(1+ni) \cong \mathbf{Z}/a(n)\mathbf{Z}$.

Q 7. Let p be a prime number ≥ 3. Then, show that $X^p + p + 2^p$ is irreducible in $\mathbf{Q}[X]$.

OR

Prove that a value of the polynomial $X^{10} - 35X + 12$ for an integer value of X cannot be 2 or -2.

Q 8. If S is a multiplicative subset of a commutative ring A with unity, prove that the complement of S is a union of prime ideals if and only if, S satisfies the property that $ab \in S \Rightarrow a, b \in S$.

OR

If $p > 3$ is a prime, prove that p can be expressed as $a^2 + 3b^2$ for some integers a, b if and only if $p \equiv 1 \bmod 3$.